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A GEOMETRICAL INTERPRETATION OF THE POINCARÉ–CHETAYEV–RUMYANTSEV EQUATIONS†

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By introducing a tangential space to the manifold of all possible positions of a mechanical system of equations, its motions are written in the form of a single vector equation, which has the form of Newton's second law. From this equation, written for ideal non-linear time-dependent non-holonomic first-order constraints, the Poincaré–Chetayev–Rumyantsev equations, as well as other fundamental types of equations of motion, are obtained. © 2002 Elsevier Science Ltd. All rights reserved.

A number of papers by Rumyantsev [1–5], based on the Poincaré–Chetayev approach, are devoted to investigating the equations of motion of non-linear non-holonomic systems.

Suppose the motion of a free mechanical system is described, in generalized coordinates q^σ, by Lagrange's equations of the second kind

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\sigma}-\frac{\partial T}{\partial q^\sigma}=Q_\sigma,\ \ T=\frac{M}{2}g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta,\ \ q^0=t,\ \ \dot{q}^0=1 \tag{1}$$

where Q_σ is the generalized force corresponding to the coordinate q^σ, and M is the mass of the whole system. Here and henceforth, summation over repeated sub- and/or superscripts is assumed; the indices ρ, σ, τ and ε take the values $1, 2, \ldots, s$; α, β, γ and δ take the values $0, 1, 2, \ldots, s$; $\varkappa$ and ν take the values $1, 2, \ldots, k$; λ and μ take the values $1, 2, \ldots, 1$, where $l = s - k$.

We will introduce a manifold of all those positions of the mechanical system in question which it can have at a given instant of time t. We will fix a certain point of this manifold, specified by the coordinates q^σ. Suppose the old and new coordinates of this point are expressed in terms of one another by the formulae

$$q^\sigma = q^\sigma(t, q_*),\ \ q_*^\rho = q_*^\rho(t, q)$$

or, in differential form

$$\delta q^\sigma = \frac{\partial q^\sigma}{\partial q_*^\rho}\delta q_*^\rho,\ \ \delta q_*^\rho = \frac{\partial q_*^\rho}{\partial q^\sigma}\delta q^\sigma$$

The quantities δq^σ and δq_*^ρ, connected by these relations, are called the contravariant components of the tangential vector $\delta\mathbf{y}$, and the whole set of vectors $\delta\mathbf{y}$ is called a tangential space to the manifold introduced above at this point [6]. The vector $\delta\mathbf{y}$ can be conveniently represented in the form

$$\delta\mathbf{y} = \delta q^\sigma \mathbf{e}_\sigma$$

while the set of vectors $\mathbf{e}_\sigma$ is considered as the basis of the tangential space in the system of coordinates q^σ. We will introduce a Euclidean structure in the tangential space using the invariance of the positive-definite quadratic form

$$(\delta\mathbf{y})^2 = g_{\sigma\tau}\delta q^\sigma \delta q^\tau = g_{\rho\varepsilon}^*\delta q_*^\rho \delta q_*^\varepsilon$$

Here $g_{\sigma\tau}$ and $g_{\rho\varepsilon}^*$ are coefficients which occur in the expression for the kinetic energy in coordinates q^σ and q_*^ρ respectively.

The generalized forces Q_σ, which occur in system of equations (1), are, by definition, the coefficients of the variations of the coordinates δq^σ in the expression for the possible elementary work δA. Using

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the continuous numbering $i = 1, 2, 3, \dots$ for the notation both of the Cartesian coordinates x_i of the points of application of the forces and for the projections X_i of these forces, we can write

$$\delta A = X_i \delta x_i$$

Taking into account the fact that

$$\delta x_i = \frac{\partial x_i}{\partial q^\sigma}\delta q^\sigma = \frac{\partial x_i}{\partial q_*^\rho}\delta q_*^\rho$$

we obtain

$$\delta A = Q_\sigma \delta q^\sigma = Q_\rho^* \delta q_*^\rho \tag{2}$$

where

$$Q_\sigma = X_i \frac{\partial x_i}{\partial q^\sigma},\ \ Q_\rho^* = X_i \frac{\partial x_i}{\partial q_*^\rho} = Q_\sigma \frac{\partial q^\sigma}{\partial q_*^\rho}$$

Expression (2) is a linear invariant differential form of the vector $\delta \mathbf{y}$. Its coefficients Q_σ and Q_ρ^* when using the coordinates q^σ and q_*^ρ respectively are the components of the covector $\mathbf{Y}$ [6].

Using the Euclidean structure of the tangential space, we will represented δA in the form of a scalar product

$$\delta A = \mathbf{Y} \cdot \delta \mathbf{y},\ \ \mathbf{Y} = Q_\sigma \mathbf{e}^\sigma$$

Here $\mathbf{e}^\sigma$ are the vectors of the mutual basis, given by the relations

$$\mathbf{e}^\sigma \cdot \mathbf{e}_\tau = \delta_\tau^\sigma = \begin{cases} 0, & \sigma \neq \tau \\ 1, & \sigma = \tau \end{cases}$$

Hence, from the expressions $g_{\sigma\tau} = \mathbf{e}_\sigma \cdot \mathbf{e}_\tau$ it follows that

$$\mathbf{e}_\tau = g_{\sigma\tau}\mathbf{e}^\sigma,\ \ \mathbf{e}^\sigma = g^{\sigma\tau}\mathbf{e}_\tau$$

The coefficients $g^{\sigma\tau}$ are the elements of the matrix inverse to the matrix with elements $g_{\sigma\tau}$.

The introduction of the covector $\mathbf{Y}$ in the expression for the possible elementary work δA enables us to consider the system of equations (1) as a single vector equality

$$M\mathbf{W} = \mathbf{Y}$$

Here

$$\begin{gathered} \mathbf{W} = W_\sigma \mathbf{e}^\sigma = \frac{1}{M}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\sigma} - \frac{\partial T}{\partial q^\sigma}\right)\mathbf{e}^\sigma = (g_{\sigma\tau}\ddot{q}^\tau + \Gamma_{\sigma,\alpha\beta}\dot{q}^\alpha \dot{q}^\beta)\mathbf{e}^\sigma = W^\sigma \mathbf{e}_\sigma = \\ = (\ddot{q}^\sigma + \Gamma_{\alpha\beta}^\sigma \dot{q}^\alpha \dot{q}^\beta)\mathbf{e}_\sigma \\ \Gamma_{\alpha\beta}^\sigma = g^{\sigma\tau}\Gamma_{\tau,\alpha\beta} = \frac{1}{2}g^{\sigma\tau}\left(\frac{\partial g_{\tau\beta}}{\partial q^\alpha} + \frac{\partial g_{\tau\alpha}}{\partial q^\beta} - \frac{\partial g_{\alpha\beta}}{\partial q^\tau}\right) \end{gathered} \tag{3}$$

We will now investigate the non-free motion. By the constraint elimination principle, the application of forces leads to the occurrence of a reaction $\mathbf{R}$, and hence we will have

$$M\mathbf{W} = \mathbf{Y} + \mathbf{R} \tag{4}$$

Consider non-linear time-dependent non-holonomic first-order constraints, specified in the form

$$f_1^\varkappa(t, q, \dot{q}) = 0 \tag{5}$$

Differentiating these constraints with respect to time, we obtain

$$f_2^{\varkappa}(t,q,\dot q,\ddot q) \equiv a_{2\sigma}^{l+\varkappa}(t,q,\dot q)\ddot q^{\sigma} + a_{20}^{l+\varkappa}(t,q,\dot q) = 0, \quad l = s-k \tag{6}$$

Note that linear non-holonomic second-order constraints can also be specified in this form. Holonomic constraints lead to relations (6) after double differentiation with respect to time.

The introduction of a tangential space and of the vector $\mathbf{W}$, specified by formula (3), enables us to write the system of equations (6) in vector form

$$\mathbf{a}^{l+\varkappa}\cdot\mathbf{W} = \chi_2^{\varkappa}(t,q,\dot q)$$

$$\mathbf{a}^{l+\varkappa} = a_{2\sigma}^{l+\varkappa}\mathbf{e}^{\sigma}, \quad \chi_2^{\varkappa} = -a_{20}^{l+\varkappa} + a_{2\sigma}^{l+\varkappa}\Gamma_{\alpha\beta}^{\sigma}\dot q^{\alpha}\dot q^{\beta} \tag{7}$$

It can be seen from these expressions that when there are constraints in s-dimensional tangential space it is best to introduce into consideration a subspace, the basis of which is the vectors $\mathbf{a}^{l+\varkappa}$ (K-space). Then, the whole space can be represented in the form of the direct sum of this space and its orthogonal complement with basis $\mathbf{a}_{\lambda}$ (L-space), where

$$\mathbf{a}_{\lambda}\cdot\mathbf{a}^{l+\varkappa} = 0$$

Note that this subdivision of the tangential space by constraint equations corresponds to fixed values of the variables t, q^{σ}, $\dot q^{\sigma}$.

It was shown in (7), that the component $\mathbf{W}^{K}$ of the vector $\mathbf{W}$, belonging to K-space, is completely defined by the constraint equations (7), if

$$|h^{\varkappa\nu}| \neq 0 \tag{8}$$

where

$$h^{\varkappa\nu} = \mathbf{a}^{l+\varkappa}\cdot\mathbf{a}^{l+\nu}$$

It has also been shown that in the case of ideal constraints, Eq. (4) takes the form

$$M\mathbf{W} = \mathbf{Y} + \Lambda_{\varkappa}\mathbf{a}^{l+\varkappa} \tag{9}$$

Thus, vector equation (8) expresses the law of motion of both holonomic and non-holonomic systems, in which, when there are ideal constraints, the generalized accelerations $\ddot q^{\sigma}$ satisfy system of equations (6), while the vectors $\mathbf{a}^{l+\varkappa}$ satisfy condition (8). It is essential that this equation has a vector form, invariant to the choice of the system of coordinates in which the constraint equations are specified and in which the motion is described. Hence, we obtain from this the fundamental forms of the equations of motion of non-holonomic systems and we thereby show their equivalence.

Projecting Eq. (9) onto any constructed system of vectors $\mathbf{a}_{\lambda}$, forming the basis of L-space, we obtain the following system of scalar equations

$$M\mathbf{W}\cdot\mathbf{a}_{\lambda} = \mathbf{Y}\cdot\mathbf{a}_{\lambda} \tag{10}$$

Supplementing Eqs (10) by Eqs (7), we obtain a closed system of equations, which enables us to obtain the law of motion in the form

$$\mathbf{W} = \mathbf{F}(t,q,\dot q)$$

The reduction of the problem to this equation can be regarded, by Novoselov's expression [8, p.28], as "the reduction of the problem of non-holonomic mechanics to a conventional problem of the mechanics of holonomic systems".

The specific form of Eqs (10) depends both on the method by which the system of vectors $\mathbf{a}_{\lambda}$ is specified and on the form in which the scalar products $M\mathbf{W}\cdot\mathbf{a}_{\lambda}$ are expanded. We will consider the fundamental forms of Eqs (10).

The integrable differential constraints and first-order linear non-holonomic constraints will be regarded as special cases of the constraints specified by Eqs (5). By assumption, the vectors $\mathbf{a}^{l+\varkappa}$ satisfy condition (8) and hence it follows from the constraint equations that the generalized velocities $\dot q^{\sigma}$ for specified values of the variables t and q^{σ} can be expressed in terms of the independent variables

v_*^λ. In [8–10] they are called kinematic characteristics, while in [1–5, 11–17], devoted to the Poincaré–Chetayev equations, they are called Poincaré parameters. The variables v_*^λ are given by the functions

$$v_*^\lambda = f_*^\lambda(t, q, \dot{q})$$

supplementing which by the functions

$$v_*^{l+\varkappa} = f_*^{l+\varkappa}(t, q, \dot{q}) = f_1^\varkappa(t, q, \dot{q})$$

we will have

$$\dot{q}^\sigma = \dot{q}^\sigma(t, q, v_*) \tag{11}$$

Suppose at least one of these expressions $f_*^\sigma dt$ is not a total differential or cannot be reduced to it. Then, as is well known, the variables

$$\pi^\sigma = \int_{t_0}^{t} v_*^\sigma(t)dt$$

cannot be regarded as a new system of Lagrangian coordinates. They are therefore called quasi-coordinates, and the quantities $\dot{\pi}^\sigma = v_*^\sigma$ are called quasi-velocities. For linear constraints the generalized velocities and quasi-velocities (Poincaré parameters) are connected by the relations

$$v_*^\rho = a_\sigma^\rho(t, q)\dot{q}^\sigma + a_0^\rho(t, q), \quad \dot{q}^\sigma = b_\tau^\sigma(t, q)v_*^\tau + b_0^\sigma(t, q)$$

or in abbreviated form

$$v_*^\alpha = a_\beta^\alpha \dot{q}^\beta, \quad \dot{q}^\alpha = b_\beta^\alpha v_*^\beta, \quad q^0 = t, \quad \dot{q}^0 = v_*^0 = 1, \quad a_\beta^0 = b_\beta^0 = \delta_\beta^0 \tag{12}$$

The use of the variables v_*^ρ enables us to introduce the vectors

$$\mathbf{a}^\alpha = \frac{\partial v_*^\rho}{\partial \dot{q}^\sigma}\mathbf{e}^\sigma, \quad \mathbf{a}_\tau = \frac{\partial \dot{q}^\sigma}{\partial v_*^\tau}\mathbf{e}_\sigma$$

such that

$$\mathbf{a}^\rho \cdot \mathbf{a}_\tau = \delta_\tau^\rho \tag{13}$$

A system of vectors $\mathbf{a}_\lambda$ then forms a basis of L-space, since

$$\mathbf{a}^{l+\varkappa} = \frac{\partial v_*^{l+\varkappa}}{\partial \dot{q}^\sigma}\mathbf{e}^\sigma, \quad \mathbf{a}^{l+\varkappa} \cdot \mathbf{a}_\lambda = 0$$

Consideration of the constraint equations by representing the generalized velocities in the form

$$\dot{q}^\sigma = F^\sigma(t, q, v_*^1, \ldots, v_*^l)$$

in accordance with Rumyantsev's expression [2, p. 3] means that "the parametrization of the constraints imposed on the system has been carried out". When this is so the basis of the L-space becomes known and is specified by the formulae

$$\mathbf{a}_\lambda = \frac{\partial F^\sigma}{\partial v_*^\lambda}\mathbf{e}_\sigma$$

Hence, the splitting of the tangential space by the constraint equations into K and L subspaces can be achieved by the parametrization. Then, the basis of the L-space is known, which is also exactly necessary in order to change to the specific form of writing Eqs (10).

If the constraints are linear, their parametrization, as follows from (12), will be

$$\dot q^\sigma = b_\lambda^\sigma(t,q)v_*^\lambda + b_0^\sigma(t,q)$$

Consequently, in this case we have

$$\mathbf{a}_\lambda = b_\lambda^\sigma(t,q)\mathbf{e}_\sigma$$

The vector $M\mathbf{W}$, which occurs in Eq. (10), can be represented in the form

$$M\mathbf{W} = \frac{d(M\mathbf{V})}{dt};\quad M\mathbf{V} = \frac{\partial T}{\partial \dot q^\sigma}\mathbf{e}^\sigma,\quad M\mathbf{V} = \frac{\partial T}{\partial \dot q^\sigma}\frac{\partial \dot q^\sigma}{\partial v_*^\rho}\mathbf{a}^\rho \tag{14}$$

The generalized velocities $\dot q^\sigma$ will be considered to be functions of all the variables v_*^ρ, and only in the final expressions, taking the constraint equations into account, will we assume $v_*^{l+\varkappa} = 0$. With this approach we will have

$$M\mathbf{V} = \frac{\partial T}{\partial \dot q^\sigma}\frac{\partial \dot q^\sigma}{\partial v_*^\rho}\mathbf{a}^\rho = \frac{\partial T^*}{\partial v_*^\rho}\mathbf{a}^\rho \tag{15}$$

where $T^* = T^*(t, q, v_*)$ are functions of the variables t, q^σ and v_*^σ, obtained by substituting representations (11) into the expression for the kinetic energy $T = T(t, q, \dot q)$.

If follows from (14) and (15) that

$$M\mathbf{W}\cdot\mathbf{a}_\lambda = \frac{d}{dt}\frac{\partial T}{\partial v_*^\lambda} + \frac{\partial T^*}{\partial v_*^\rho}\dot{\mathbf{a}}^\rho\cdot\mathbf{a}_\lambda$$

Taking expressions (13) into account we obtain

$$\dot{\mathbf{a}}^\rho\cdot\mathbf{a}_\lambda = -\mathbf{a}^\rho\cdot\dot{\mathbf{a}}_\lambda$$

and, consequently

$$M\mathbf{W}\cdot\mathbf{a}_\lambda = \frac{d}{dt}\frac{\partial T^*}{\partial v_*^\lambda} - M\mathbf{V}\cdot\dot{\mathbf{a}}_\lambda \tag{16}$$

Since

$$\dot{\mathbf{a}}_\lambda = \left(\frac{d}{dt}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\right)\mathbf{e}_\sigma + \frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\dot{\mathbf{e}}_\sigma,\quad \mathbf{e}_\sigma = \frac{\partial \mathbf{V}}{\partial \dot q^\sigma},\quad \dot{\mathbf{e}}_\sigma = \frac{\partial \mathbf{V}}{\partial q^\sigma}$$

$$\frac{\partial T^*}{\partial q^\tau} = \frac{\partial T}{\partial q^\tau} + \frac{\partial T}{\partial \dot q^\sigma}\frac{\partial \dot q^\sigma}{\partial q^\tau}$$

we have

$$M\mathbf{V}\cdot\dot{\mathbf{a}}_\lambda = \frac{\partial T}{\partial \dot q^\sigma}\left(\frac{d}{dt}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda} - \frac{\partial \dot q^\tau}{\partial v_*^\lambda}\frac{\partial \dot q^\sigma}{\partial q^\tau}\right) + \frac{\partial \dot q^\tau}{\partial v_*^\lambda}\frac{\partial T^*}{\partial q^\tau} \tag{17}$$

It follows from (16) and (17) that Eqs (10) can be represented in the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial v_*^\lambda} - \frac{\partial T^*}{\partial \pi^\lambda} - \frac{\partial T}{\partial \dot q^\sigma}T_\lambda^\sigma = \tilde Q_\lambda \tag{18}$$

where

$$\frac{\partial}{\partial \pi^\lambda} = \frac{\partial \dot q^\tau}{\partial v_*^\lambda}\frac{\partial}{\partial q^\tau},\quad T_\lambda^\sigma = \frac{d}{dt}\frac{\partial \dot q^\sigma}{\partial v_*^\lambda} - \frac{\partial \dot q^\sigma}{\partial \pi^\lambda},\quad \tilde Q_\lambda = Q_\sigma\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}$$

and $\tilde Q_\lambda$ are generalized forces corresponding to the Poincaré parameters (the quasi-velocities) v_*^λ.

Taking into account the fact that

$$\frac{\partial T^*}{\partial v_*^\rho}\frac{\partial v_*^\rho}{\partial \dot q^\sigma}=\frac{\partial T}{\partial \dot q^\sigma}$$

we can write Eqs (18) in the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial v_*^\lambda}-\frac{\partial T^*}{\partial \pi^\lambda}+\frac{\partial T^*}{\partial v_*^\rho}W_\lambda^\rho=\tilde Q_\lambda;\quad W_\lambda^\rho=-\frac{\partial v_*^\rho}{\partial \dot q^\sigma}T_\lambda^\sigma \tag{19}$$

Equations (18) and (19), as follows from their derivation, can be applied to both holonomic and non-holonomic systems, with ideal constraints that are both linear and non-linear with respect to the velocities. Equations (18) and (19) were obtained by Hamel [18] in 1938 for the case when the time does not occur explicitly in the kinetic energy or in the constraint equations, and were obtained by Novoselov [9] in 1957 for the general case. In 1998 Rumyantsev [4] obtained these equations by generalizing Poincaré's and Chetayev's equations. He established [5, p. 57], that these equations "... can be regarded as general equations of classical mechanics, including all known equations of motion as special cases".

As Hamel and Novoselov showed, the coefficients W_λ^ρ can be converted to the form

$$W_\lambda^\rho=\frac{\partial \dot q^\sigma}{\partial v_*^\lambda}\left(\frac{d}{dt}\frac{\partial v_*^\rho}{\partial \dot q^\sigma}-\frac{\partial v_*^\rho}{\partial q^\sigma}\right) \tag{20}$$

In the case of linear homogeneous time-independent constraints, the coefficients W_λ^ρ, as follows from (12) and (20), are

$$W_\lambda^\rho=c_{\lambda\mu}^\rho v_*^\mu,\quad c_{\lambda\mu}^\rho=\left(\frac{\partial a_\sigma^\rho}{\partial q^\tau}-\frac{\partial a_\tau^\rho}{\partial q^\sigma}\right)b_\lambda^\sigma b_\mu^\tau \tag{21}$$

Equations (19) in this case take the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial v_*^\lambda}-\frac{\partial T^*}{\partial \pi^\lambda}+c_{\lambda\mu}^\rho v_*^\mu\frac{\partial T^*}{\partial v_*^\rho}=\tilde Q_\lambda \tag{22}$$

For the case when $l = s$, these equations, and also the expressions for the coefficients $c_{\sigma\tau}^\rho$, as pointed out by Novoselov [9, p. 55], "... were first obtained by Voronets [19] in 1901 and these results were again obtained by Hamel [20] in 1904". Further, Novoselov writes that in 1901 several earlier results by Voronets appeared in a note by Poincaré [21], who obtained equations extremely close to Eq. (22). Poincaré's equations correspond to the case when the coefficients $c_{\sigma\tau}^\rho$ are constant in Eqs (22) for $l = s$, and the forces are expressed in terms of the force function

$$\tilde Q_\tau=b_\tau^\sigma\frac{\partial U}{\partial q^\sigma}$$

Equations (22), therefore, can be written in the form proposed by Poincaré [13]

$$\begin{aligned}&\frac{d}{dt}\frac{\partial L^*}{\partial v_*^\tau}=c_{\sigma\tau}^\rho v_*^\sigma\frac{\partial L^*}{\partial v_*^\rho}+X_\tau L^*;\quad L^*(q,v_*)=T^*+U,\\ &X_\tau=b_\tau^\sigma\frac{\partial}{\partial q^\sigma}\end{aligned} \tag{23}$$

Here $L^*(q, v_*)$ is Lagrange's function while X_τ are linear differential operators, which form a basis of a certain s-dimensional Lie algebra [13, p. 43] with commutator

$$[X_\sigma,X_\tau]=X_\sigma X_\tau-X_\tau X_\sigma=c_{\sigma\tau}^\rho X_\rho \tag{24}$$

where $c_{\sigma\tau}^\rho$ are structural constants of Lie algebra. It was noted in [13] that an arbitrarily chosen system of s operators acting in s-dimensional space, for which only the condition det $[b_\tau^\sigma(q)] \neq 0$

is satisfied, does not form a Lie algebra, since in this case the coefficients $c^{\rho}_{\sigma\tau}$ in (24) will be functions of q^{σ}.

The use of a tangential space and the vectors $\mathbf{a}^{\rho}$ and $\mathbf{a}_{\tau}$ in it enable the last expression in (23) and Eq. (24) to be written respectively in the form

$$X_{\tau} = \mathbf{a}_{\tau} \cdot \nabla$$

$$[X_{\sigma}, X_{\tau}] = \mathbf{a}_{\sigma} \cdot \nabla\, (\mathbf{a}_{\tau} \cdot \nabla) - \mathbf{a}_{\tau} \cdot \nabla\, (\mathbf{a}_{\sigma} \cdot \nabla) = c^{\rho}_{\sigma\tau}\mathbf{a}_{\rho} \cdot \nabla = c^{\rho}_{\sigma\tau} X_{\rho}$$

It follows from this representation of the operators X_{τ} and their commutator that they also form a closed system of operators [4] for variable coefficients $c^{\rho}_{\sigma\tau}$.

We will now introduce contravariant components of the vector $\delta\mathbf{y}$ in the basis $\{\mathbf{a}_{\tau}\}$, denoting them by $\delta' v^{\rho}_{*}$, i.e. assuming

$$\delta' v^{\rho}_{*} = \delta\mathbf{y} \cdot \mathbf{a}^{\rho}$$

In this case will have

$$\delta\mathbf{y} = \delta' v^{\tau}_{*}\mathbf{a}_{\tau} = \delta' v^{\tau}_{*} b^{\sigma}_{\tau}\mathbf{e}_{\sigma} = \delta q^{\sigma}\mathbf{e}_{\sigma}$$

and, consequently

$$\delta q^{\sigma} = b^{\sigma}_{\tau}\delta' v^{\tau}_{*}$$

Suppose $\mathbf{r}(t, q)$ is the radius vector of an arbitrary point of the mechanical system. Then

$$\delta\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^{\sigma}}\delta q^{\sigma} = b^{\sigma}_{\tau}\frac{\partial \mathbf{r}}{\partial q^{\sigma}}\delta' v^{\tau}_{*} = \delta' v^{\tau}_{*} x_{\tau}\mathbf{r} \tag{25}$$

The operators x_{τ} hence enable the possible displacements $\delta\mathbf{r}$, occurring in the general equation of mechanics, to be represented in the form (25).

Poincaré made a surprising discovery. He established that mechanical systems exist in which the tangential space possesses a remarkable property. The basis $\mathbf{a}_{\tau} = b^{\sigma}_{\tau}\mathbf{e}_{\sigma}$ introduced in it, corresponding to quasi-velocities, is specified by those functions b^{σ}_{τ} of the generalized coordinates for which the coefficients $c^{\rho}_{\sigma\tau}$ in commutator (24) are constant. As already noted, the operators x_{τ} then form a basis of a Lie algebra. A characteristic example of a mechanical system with such a property of the tangential space is an absolutely rigid body rotating around a fixed point. The Poincaré parameters in this case are, in particular, the projections of the vector of the instantaneous angular velocity onto the principle axes of inertia. Poincaré's equations (23) in this case become the dynamic Euler equations (see, for example, [2]).

We will consider the case when the linear transformations (12) are inhomogeneous and time-dependent. Equations (19) in this case, as follows from (20) and (21), when there are both potential and non-potential forces, can be written in the form

$$\frac{d}{dt}\frac{\partial L^{*}}{\partial v^{\lambda}_{*}} - \frac{\partial L^{*}}{\partial \pi^{\lambda}} = c^{\rho}_{\mu\lambda} v^{\mu}_{*}\frac{\partial L^{*}}{\partial v^{\rho}_{*}} + c^{\rho}_{0\lambda}\frac{\partial L^{*}}{\partial v^{\rho}_{*}} + \tilde{Q}_{\lambda} \tag{26}$$

Here

$$c^{\rho}_{\alpha\beta} = a^{\rho}_{\gamma}\left(\frac{\partial b^{\gamma}_{\beta}}{\partial q^{\delta}}b^{\delta}_{\alpha} - \frac{\partial b^{\gamma}_{\alpha}}{\partial q^{\delta}}b^{\delta}_{\beta}\right) = \left(\frac{\partial a^{\rho}_{\gamma}}{\partial q^{\delta}} - \frac{\partial a^{\rho}_{\delta}}{\partial q^{\gamma}}\right)b^{\gamma}_{\alpha}b^{\delta}_{\beta}$$

Two different representations of the coefficients $c^{\rho}_{\alpha\beta}$ follow from the fact that

$$a^{\rho}_{\gamma}b^{\gamma}_{\beta} = \delta^{\rho}_{\beta}$$

Equations (26) are called the equations of non-holonomic systems in Poincaré–Chetayev variables [2, 13, 15], and also the equations of motion of non-holonomic systems in quasi-coordinates [22, 23]. Chetayev extended Poincaré's equations (23) to the case when the number of Lagrange coordinates is greater than the number of independent Poincaré parameters, i.e. he obtained, using the Poincaré approach, Eqs (22) for the case when the coefficients $c_{\lambda\mu}^{l+\varkappa}=0$ while the coefficients $c_{\lambda\mu}^{\nu}$ are constant. He pointed out, however, that the equations he obtained also make sense for variables coefficients $c_{\lambda\mu}^{\nu}$ [12]. This extension of Poincaré's equations is carried out in [1–5, 14–17].

In conclusion we will consider the two simplest forms of the expansion of the scalar products in Eqs (10), proposed by Appele and Maggi.

Introducing Appele's function

$$T_1 = M\mathbf{W}^2/2$$

we can write

$$M\mathbf{W} = MW_\sigma \mathbf{e}^\sigma = \frac{\partial T_1}{\partial \ddot{q}^\sigma}\mathbf{e}^\sigma = \frac{\partial T_1^*}{\partial \dot{v}_*^\rho}\mathbf{a}^\rho$$

Hence, using Eqs (10) we arrive at Appele's equations

$$\frac{\partial T_1^*}{\partial \dot{v}_*^\lambda} = \tilde{Q}_\lambda$$

We obtain Maggi's equations

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\sigma} - \frac{\partial T}{\partial q^\sigma} - Q_\sigma\right)\frac{\partial \dot{q}^\sigma}{\partial v_*^\lambda} = 0$$

from Eqs (10) if we use expression (3) and also the fact that

$$\mathbf{a}_\lambda = \frac{\partial \dot{q}^\sigma}{\partial v_*^\lambda}\mathbf{e}_\sigma$$

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